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STRESS FUNCTIONS FOR COSSERAT ELASTICITY

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Abstract—In this paper a complete solution is obtained to the displacement equations of equilibrium for linear infinitesimal isotropic Cosserat elasticity. This solution is in terms of stress functions analogous to the Papkovitch functions of classical elasticity and those found by Mindlin and Tiersten for couple stress elasticity. Previous complete solutions to these equations given by Mindlin and by Neuber contain three more scalar potentials than are employed in the present solution.

1. INTRODUCTION

THE equations for the theory of the linear infinitesimal isotropic elastic Cosserat continuum were presented by Aero and Kuvshinskii [1, 2], Mindlin [3] and Neuber [4]. The Cosserat theory may be considered as a special case of the more general theories of Green and Rivlin [5], Mindlin [6] and Toupin [7]. The linear theory of micropolar elasticity recently proposed by Eringen [8] coincides with the Cosserat theory. The special case of Cosserat elasticity known as couple stress elasticity was presented in the papers of Aero and Kuvshinskii [9], Grioli [10], Mindlin [11] and Mindlin and Tiersten [12].

In the next section the equations for the Cosserat theory are reviewed. In the following and final section a complete solution is obtained for the displacement equations of equilibrium for a linear infinitesimal isotropic Cosserat elasticity. This solution is very similar in form to that obtained by Mindlin and Tiersten [12] for the couple stress theory. The previous complete solutions given by Mindlin [3] and by Neuber [4] contain three more scalar potentials than are employed here. The method of proof employed here differs only in algebraic detail from that employed by Mindlin [3, 13].

Notation

Cartesian tensor notation will be employed. The symmetric part $\tau_{(ij)}$ of τ_{ij} and the skew symmetric part $\sigma_{[ij]}$ of σ_{ij} are given by the formulas

$$\tau_{(ij)} \equiv \tfrac{1}{2}(\tau_{ij}+\tau_{ji}), \qquad \sigma_{[ij]} \equiv \tfrac{1}{2}(\sigma_{ij}-\sigma_{ji}). \tag{1.1}$$

The following rule is laid down for associating axial tensors with absolute three dimensional tensors of rank two and three: Given a skew symmetric second rank tensor $\psi_{[jk]}$ and a third rank tensor skew symmetric in its last two indices $\kappa_{m[jk]}$, the axial vector denoted by $\hat{\psi}_i$ and the axial second rank tensor denoted by $\hat{\kappa}_{mi}$ are computed by the formulas

$$\hat{\psi}_i \equiv \tfrac{1}{2}\, e_{ijk}\psi_{jk}, \qquad \hat{\kappa}_{mi} \equiv \tfrac{1}{2}\, e_{ijk}\kappa_{mjk}. \tag{1.2}$$

The inversions of equations (1.1) are given by

$$\psi_{[jk]} = e_{ijk}\hat{\psi}_i, \qquad \kappa_{m[jk]} = e_{ijk}\hat{\kappa}_{mi}. \tag{1.3}$$

An axial or pseudo scalar will be denoted by a symbol with a circumflex or hat on top and no indices, for example, $\hat{\theta}$ or $\hat{h}$.

Generally the notation employed here follows that of Mindlin [3, 6] and Mindlin and Tiersten [12]. A table giving the notations of Mindlin [3], Aero and Kuvshinskii [1, 2], Neuber [4] and Eringen [8] is included as an appendix.

2. THE COSSERAT THEORY

Kinematics

In Cosserat elasticity, in addition to the displacement vector u_i of classical elasticity, an axial vector $\hat{\psi}_i$ is introduced to represent the *total rotation* of the rigid Cosserat triad during deformation. The vector fields u_i and $\hat{\psi}_i$ are the basic kinematical quantities in the Cosserat theory. The usual strain tensor ε_{ij} and the usual rotation tensor ω_{ij} are defined by

$$\varepsilon_{ij} \equiv u_{(i,j)}, \qquad \omega_{ij} \equiv u_{[i,j]}. \tag{2.1}$$

The traditional *average rotation* tensor ω_{ij} is a measure of the rotation of principal axes of strain during deformation; the associated axial vector is given by

$$\hat{\omega}_i = \tfrac{1}{2}\, e_{ijk} u_{j,k}, \qquad \hat{\boldsymbol{\omega}} = -\tfrac{1}{2}\nabla x\mathbf{u}. \tag{2.2}$$

The *relative rotation* between the triad and the principal axes of strain is then defined as γ_{ij}

$$\gamma_{ij} \equiv \omega_{ji} - \psi_{ij} = -\gamma_{ji}, \tag{2.3}$$

which is also represented by an axial vector

$$\hat{\gamma}_k = -\hat{\omega}_k - \hat{\psi}_k, \qquad \hat{\boldsymbol{\gamma}} = \tfrac{1}{2}\nabla x\mathbf{u} - \hat{\boldsymbol{\psi}}. \tag{2.4}$$

The *gradient of the total rotation* is denoted by $\hat{\kappa}_{im}$

$$\hat{\kappa}_{im} = \hat{\psi}_{m,i} \quad \text{or} \quad \kappa_{ijk} \equiv \psi_{jk,i}. \tag{2.5}$$

Note from (2.4) and (2.5) that

$$\hat{\kappa}_{mm} = -\hat{\gamma}_{m,m}, \tag{2.6}$$

hence when the relative rotation $\hat{\gamma}_m$ vanishes everywhere as it does in the couple stress theory, $\hat{\kappa}_{mm}$ vanishes also.

Equilibrium equations

If the *symmetric stress tensor* is denoted by τ_{ij}, the *skew symmetric stress tensor* by σ_{ij}, and the *couple stress tensor* by $\hat{\mu}_{ij}$, then the equations of equilibrium (cf. Mindlin [6], Sections 3 and 4) are given by

$$\tau_{ji,j} + \sigma_{ji,j} + f_i = 0 \tag{2.7}$$

and

$$2\hat{\mu}_{ji,j} + 2\hat{\sigma}_i + \hat{c}_i = 0, \tag{2.8}$$

where f_i is the *body force per unit volume* and $\hat{c}_i$ is the *body couple per unit volume*. The *traction boundary conditions* are

$$t_j = n_i\tau_{ij} + n_i\sigma_{ij}, \qquad \hat{m}_j = n_i\hat{\mu}_{ij}, \tag{2.9}$$

where t_j is the *surface force per unit area*, $\hat{m}_j$ is the *surface couple per unit area*, and n_i is the *unit normal* to the boundary.

Constitutive equations

The constitutive equations for linear infinitesimal isotropic Cosserat elasticity are

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \qquad \sigma_{ij} = 2\tau\gamma_{ij}, \tag{2.10}$$

$$\hat{\mu}_{ij} = \frac{\alpha}{2}\delta_{ij}\hat{\kappa}_{kk} + 2\eta\hat{\kappa}_{ij} + 2\eta'\hat{\kappa}_{ji} \tag{2.11}$$

where λ and μ are the Lamé coefficients of classical elasticity and η, η' are moduli employed in the couple stress theory (cf. equation (3.23) of Mindlin and Tiersten [12]). The coefficients τ and α appear in neither classical elasticity nor the couple stress theory; τ is a *modulus of local rotational stiffness* and α is a *modulus of the volume flux of local rotation.* It is required that $3\lambda+2\mu$, μ, η, τ, $3\alpha+4\eta+4\eta'$ and the quantity $1-(\eta'/\eta)^2$ all be positive.

The coefficients μ and τ have the same dimensions and are both positive, hence the number N defined by

$$N \equiv \sqrt{\left(\frac{\tau}{\mu+\tau}\right)}, \qquad 0 \leq N \leq 1, \tag{2.12}$$

is dimensionless. N has the value 0 for classical elasticity and the value 1 for the couple stress theory. N is called the *coupling number.*

Of the six coefficients λ, μ, τ, α, η and η' introduced in (2.10) and (2.11), λ, μ and τ have the dimensions of stress and α, η and η' have the dimensions of length squared times stress, hence any ratio of α, η or η' to λ, μ or τ will be a material parameter of dimension length squared. In the couple stress theory the length l,

$$l \equiv \sqrt{\frac{\eta}{\mu}} \tag{2.13}$$

is used. In Cosserat elasticity the material lengths

$$l_1 \equiv \frac{l}{N}, \qquad l_2 \equiv \sqrt{\left(\frac{\alpha+4\eta+4\eta'}{4\tau}\right)}, \qquad l_3 \equiv \sqrt{\left[\frac{(1-N^2)}{N^2}\right]l^2} \tag{2.14}$$

are also employed. The lengths l_1, l_2 and l_3 reduce to l, 0 and 0, respectively, for the couple stress theory.

Displacement equations of equilibrium

The displacement equations of equilibrium for the Cosserat theory are obtained by substituting the constitutive equations (2.10) and (2.11) into the equations of equilibrium (2.7), (2.8) and subsequently employing the definitions (2.1) and (2.5); thus

$$(\lambda+\mu)u_{k,ki} + \mu u_{i,kk} - 2\tau e_{ijk}\hat{\gamma}_{k,j} + f_i = 0, \tag{2.15}$$

$$(\alpha+4\eta')\hat{\psi}_{k,ki} + 4\eta\hat{\psi}_{i,kk} + 4\tau\hat{\gamma}_i + \hat{c}_i = 0. \tag{2.16}$$

These two equations may be rewritten in vector notation as

$$(\lambda+2\mu)\nabla\nabla\,.\,\mathbf{u}-\mu\nabla\times\nabla\times\mathbf{u}-2\tau\nabla\times\hat{\boldsymbol{\gamma}}+\mathbf{f}=0, \tag{2.17}$$

$$(\alpha+4\eta+4\eta')\nabla\nabla\,.\,\hat{\boldsymbol{\psi}}-4\eta\nabla\times\nabla\times\hat{\boldsymbol{\psi}}+4\tau\hat{\boldsymbol{\gamma}}+\hat{\mathbf{c}}=0, \tag{2.18}$$

where the vector identity

$$\nabla^2\mathbf{v}=\nabla\nabla\,.\,\mathbf{v}-\nabla\times\nabla\times\mathbf{v} \tag{2.19}$$

has been employed. The system of 3 vector equations consisting of (2.17), (2.18) and (2.4) involve the three vector unknowns $\mathbf{u}$, $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\psi}}$.

3. DISPLACEMENT POTENTIALS

For a regular region of space a complete solution to the displacement equations of equilibrium (2.17) and (2.18) is given by

$$\mathbf{u}=\mathbf{h}-l^2\nabla\nabla\,.\,\mathbf{h}-l^4\frac{(1-N^2)}{N^4}\nabla^2\nabla\nabla\,.\,\mathbf{h}-\frac{1}{4(1-\nu)}\nabla\left[\mathbf{r}\,.\left(1-\frac{l^2}{N^2}\nabla^2\right)\mathbf{h}+h_0\right] \tag{3.1}$$

$$\hat{\boldsymbol{\gamma}}=\nabla\hat{h}-\frac{(1-N^2)}{2N^4}l^2\nabla\times\nabla^2\mathbf{h}-\frac{(1-N^2)}{N^2}\frac{l^2}{2\tau}\nabla\times\mathbf{f}-\frac{1}{4\tau}\hat{\mathbf{c}}, \tag{3.2}$$

where the potentials $\mathbf{h}$, h_0 and $\hat{h}$ satisfy the differential equations

$$\mu\left(1-\frac{l^2}{N^2}\nabla^2\right)\nabla^2\mathbf{h}=-\left[1-l^2\left(\frac{1-N^2}{N^2}\right)\nabla^2\right]\mathbf{f}-\frac{1}{2}\nabla\times\hat{\mathbf{c}}, \tag{3.3}$$

$$\mu\nabla^2h_0=\mathbf{r}\,.\left[1-\frac{(1-N^2)}{N^2}l^2\nabla^2\right]\mathbf{f}+\frac{1}{2}\mathbf{r}\,.\,\nabla\times\hat{\mathbf{c}}-\frac{(1-N^2)}{N^2}2l^2\left[1-2(1-\nu)\frac{(1-N^2)}{N^2}l^2\nabla^2\right]\nabla\,.\,\mathbf{f} \tag{3.4}$$

$$(1-l_2^2\nabla^2)\hat{h}=-\frac{l_2^2}{4\tau}\nabla\,.\,\hat{\mathbf{c}}, \tag{3.5}$$

where $\mathbf{r}$ denotes a position vector and where the notation (2.12), (2.13) and (2.14) for the material parameters has been employed. When N is set equal to one, equations (3.1), (3.3) and (3.4) reduce to equations (11.17), (11.18) and (11.19) of Mindlin and Tiersten [12] and represent a complete solution of the displacement equations of equilibrium for the couple stress theory. This result indicates that the solutions to the differential equations governing static Cosserat elasticity are no more difficult and only slightly more complicated than those governing the static couple stress elasticity. (The boundary conditions for the Cosserat theory are less complicated, however.) In the case when N is unity and $l_2=0$ it may be shown that both $\hat{\boldsymbol{\gamma}}$ and $\hat{h}$ vanish identically. On the other hand, the complete solution for the static form of the Navier equations of classical elasticity is obtained from (3.1), (3.3) and (3.4) if the body couple $\hat{\mathbf{c}}$ and the material length parameter l is set equal to zero. In this latter case equations (3.1), (3.3) and (3.4) reduce to equations (10), (11) and (12) of Mindlin [13]. Thus, the complete solution for the displacement fields in static Cosserat elasticity given above contains the complete solutions for both classical elasticity and the couple stress theory as special cases and it involves only one more scalar potential, namely $\hat{h}$, than employed in classical elasticity or the couple stress theory.

Mindlin [3] has given a solution of the displacement equations (2.17) and (2.18) involving one more vector potential than employed above. Mindlin's potentials $\mathbf{B}$, $\mathbf{K}$, K_0 and B_0 are related to $\mathbf{h}$, h_0 and $\hat{h}$ by

$$
\begin{aligned}
\mathbf{h} &= (1-l_3^2\nabla^2)\mathbf{B}+\nabla\times\mathbf{K},\\
h_0 &= (1-l_3^2\nabla^2)B_0-(1-l_1^2\nabla^2)\mathbf{r}\,.\,\nabla\times\mathbf{K}+\frac{4(1-\nu)}{\mu}l_3^4\nabla\,.\,\mathbf{f},\\
\hat{h} &= \tfrac{1}{2}\nabla\,.\,\mathbf{K}-\tfrac{1}{4}\nabla^2(\mathbf{r}\,.\,\mathbf{K}+K_0).
\end{aligned}
\tag{3.6}
$$

Neuber [4] has also given a solution of the displacement equilibrium equations in the case of vanishing body forces and vanishing body couple; Neuber's solution also employs one more vector potential than employed above. Neuber's potentials $\mathbf{\Phi}$, $\mathbf{\Psi}$, Φ_0 and N' are related to $\mathbf{h}$, h_0, $\hat{h}$ in the case of vanishing body forces and vanishing body couples by

$$
\begin{aligned}
\mathbf{h} &= \frac{4(1-\nu)}{2\mu}\mathbf{\Phi}+\frac{1}{2\mu}\mathbf{\Psi},\\
h_0 &= \frac{4(1-\nu)}{2\mu}(\Phi_0-l^2\nabla\,.\,\mathbf{\Phi}),\\
\hat{h} &= -\frac{N'}{2\tau},
\end{aligned}
\tag{3.7}
$$

where ν is Poisson's ratio. The complete solution given by equations (3.1) through (3.5) has an advantage over the solutions given by Mindlin and Neuber in that it involves only five scalar potentials while the other solutions involve eight. Also, the present solution is more easily related to similar complete solutions for the couple stress theory and for classical elasticity.

The completeness of the solution stated in the opening paragraph of this section will now be proved. The method of proof given here follows Mindlin [3, 13]. Let the Helmholtz decomposition of the fields $\mathbf{u}$ and $\hat{\gamma}$ be represented by

$$\mathbf{u} = \nabla\phi+\nabla\times\hat{\mathbf{k}}, \qquad \nabla\,.\,\hat{\mathbf{k}} = 0, \tag{3.8}$$

$$\hat{\gamma} = \nabla\hat{\theta}+\nabla\times\mathbf{q}, \qquad \nabla\,.\,\mathbf{q} = 0, \tag{3.9}$$

where ϕ and $\hat{\theta}$ represent a scalar potential and an axial or pseudoscalar potential, respectively, and $\mathbf{q}$ and $\hat{\mathbf{k}}$ represent a vector potential and an axial vector potential, respectively. Substituting (3.8) and (3.9) into (2.17) and (2.18) and employing $(2.4)_2$ one finds that

$$\nabla^2[(\lambda+2\mu)\nabla\phi+\mu\nabla\times\hat{\mathbf{k}}+2\tau\mathbf{q}]+\mathbf{f} = \mathbf{0}, \tag{3.10}$$

$$2\tau(1-l_2^2\nabla^2)\nabla\hat{\theta}+2\tau(1-l_3^2\nabla^2)\nabla\times\mathbf{q}-\eta\nabla^4\hat{\mathbf{k}}+\tfrac{1}{2}\hat{\mathbf{c}} = \mathbf{0}, \tag{3.11}$$

where the notation introduced by (2.12), (2.13) and (2.14) has been employed. Applying the operator $(1-l_3^2\nabla^2)$ to equation (3.10), taking the curl of (3.11), and subsequently equating these two expressions for the quantity $2\tau(1-l_3^2\nabla^2)\nabla^2\mathbf{q}$, one obtains the expression

$$\mu\nabla^2[k(1-l_3^2\nabla^2)\nabla\phi+(1-l_1^2\nabla^2)\nabla\times\hat{\mathbf{k}}] = -(1-l_3^2\nabla^2)\mathbf{f}-\tfrac{1}{2}\nabla\times\hat{\mathbf{c}}, \tag{3.12}$$

where the notation

$$k \equiv \frac{\lambda+2\mu}{\mu} = \frac{2(1-\nu)}{1-2\nu} \tag{3.13}$$

has been introduced. Using the customary notation of potential theory, the quantity $\mathbf{h}'_P$ is defined by

$$4\pi l_1^2 \mathbf{h}'_P \equiv \int_V \frac{1}{r_1} \mathrm{e}^{-r_1/l}[k(1-l_3^2\nabla^2)\nabla\phi + (1-l_1^2\nabla^2)\nabla\times\hat{\mathbf{k}}]_Q \, \mathrm{d}V_Q \tag{3.14}$$

where the subscript P denotes a function of a field point, the subscript Q denotes a function of the source point and r_1 denotes the distance between P and Q. Dropping the subscript P from $\mathbf{h}'_P$ it follows from (3.14) that

$$(1-l_1^2\nabla^2)\mathbf{h}' = k(1-l_3^2\nabla^2)\nabla\phi + (1-l_1^2\nabla^2)\nabla\times\hat{\mathbf{k}}, \tag{3.15}$$

and from (3.12) and (3.15)

$$\mu(1-l_1^2\nabla^2)\nabla^2\mathbf{h}' = -(1-l_3^2\nabla^2)\mathbf{f} - \tfrac{1}{2}\nabla\times\mathbf{c}. \tag{3.16}$$

Now, taking divergence of (3.10) and employing $(3.9)_2$ one obtains the equation

$$(\lambda+2\mu)\nabla^4\phi + \nabla\,.\,\mathbf{f} = 0; \tag{3.17}$$

similarly, the divergence of (3.15) yields

$$(1-l_1^2\nabla^2)\nabla\,.\,\mathbf{h}' = k(1-l_3^2\nabla^2)\nabla^2\phi. \tag{3.18}$$

From (3.18) using (3.17) and (3.13) it follows that

$$(1-l_1^2\nabla^2)\nabla\,.\,\mathbf{h}' - \frac{l_3^2}{\mu}\nabla\,.\,\mathbf{f} = k\nabla^2\phi. \tag{3.19}$$

Defining

$$2k\phi^* = \mathbf{r}\,.\,(1-l_1^2\nabla^2)\mathbf{h}' \tag{3.20}$$

it follows from (3.16) and (3.19) that

$$2k\mu\nabla^2\phi^* = 2k\mu\nabla^2\phi + 2l_3^2\nabla\,.\,\mathbf{f} - \mathbf{r}\,.\,(1-l_3^2\nabla^2)\mathbf{f} - \tfrac{1}{2}\mathbf{r}\,.\,\nabla\times\hat{\mathbf{c}}. \tag{3.21}$$

Now, define

$$h'_0 = 2k(\phi-\phi^*), \tag{3.22}$$

then from (3.21)

$$\mu\nabla^2 h'_0 = \mathbf{r}\,.\,(1-l_3^2\nabla^2)\mathbf{f} + \tfrac{1}{2}\mathbf{r}\,.\,\nabla\times\hat{\mathbf{c}} - 2l_3^2\nabla\,.\,\mathbf{f} \tag{3.23}$$

and from (3.20) and (3.22)

$$2k\phi = \mathbf{r}\,.\,(1-l_1^2\nabla^2)\mathbf{h}' + h'_0. \tag{3.24}$$

Keeping this result in mind a new calculation is begun by defining the quantity $\mathbf{g}$ by

$$\mathbf{g} \equiv \mathbf{h}' - l_1^2\nabla\nabla\,.\,\mathbf{h}' - k(1-l_3^2\nabla^2)\nabla\phi \tag{3.25}$$

which, from (3.18) is divergence free,

$$\nabla\,.\,\mathbf{g} = 0. \tag{3.26}$$

Therefore, there exists an axial vector function $\hat{\mathbf{k}}'$ such that

$$\mathbf{g} = \nabla \times \hat{\mathbf{k}}'. \tag{3.27}$$

Applying the operator $(1-l_1^2\nabla^2)$ to (3.25) it follows from (3.27), (3.15) and (3.18) that

$$(1-l_1^2\nabla^2)\nabla \times \hat{\mathbf{k}}' = (1-l_1^2\nabla^2)\nabla \times \hat{\mathbf{k}}. \tag{3.28}$$

If $\mathbf{h}''$ is defined as

$$\mathbf{h}'' \equiv \nabla \times \hat{\mathbf{k}} - \nabla \times \hat{\mathbf{k}}' \tag{3.29}$$

then from (3.28) and (3.29) one finds that

$$(1-l_1^2\nabla^2)\mathbf{h}'' = 0, \qquad \nabla . \mathbf{h}'' = 0. \tag{3.30}$$

From (3.25), (3.27) and (3.29)

$$\nabla \times \hat{\mathbf{k}} = \mathbf{h}' + \mathbf{h}'' - l_1^2\nabla\nabla . \mathbf{h}' - k(1-l_3^2\nabla^2)\nabla\phi, \tag{3.31}$$

hence introducing the notation

$$\mathbf{h} = \mathbf{h}' + \mathbf{h}'' \tag{3.32}$$

it follows from (3.24) and $(3.30)_1$ that

$$2k\phi = \mathbf{r} . (1-l_1^2\nabla^2)\mathbf{h} + h_0' \tag{3.33}$$

and from (3.31), (3.32), (3.33) and (3.30) that

$$\nabla \times \hat{\mathbf{k}} = \mathbf{h} - l_1^2\nabla\nabla . \mathbf{h} - \tfrac{1}{2}(1-l_3^2\nabla^2)\nabla[\mathbf{r} . (1-l_1^2\nabla^2)\mathbf{h} + h_0]. \tag{3.34}$$

Placing (3.33) and (3.34) into (3.8) and using (3.13) the displacement $\mathbf{u}$ is given by

$$\mathbf{u} = \mathbf{h} - l_1^2\nabla\nabla . \mathbf{h} - \frac{1}{2}\left[\frac{1}{2(1-\nu)} - l_3^2\nabla^2\right]\nabla[h_0' + \mathbf{r} . (1-l_1^2\nabla^2)\mathbf{h}]. \tag{3.35}$$

Expansion of the last term in this expression for $\mathbf{u}$ using (3.16), (3.23) and the notation (2.14) will give the result (3.1) if h_0 is given by

$$h_0 = h_0' + \frac{l_3^4}{\mu}4(1-\nu)\nabla . \mathbf{f}. \tag{3.36}$$

The differential equation (3.4) follows upon substituting (3.36) into (3.23) and employing the notation (2.14). The differential equation (3.3) follows from (3.16) when (3.30), (3.32) and (2.14) are employed.

The problem of deriving the expression (3.2) for $\hat{\gamma}$ will now be considered. Taking the curl of (3.10) it follows from $(3.8)_2$ and the vector identity (2.19) that

$$\nabla^2\nabla \times \mathbf{q} = \frac{1}{2\tau}[\mu\nabla^4\hat{\mathbf{k}} - \nabla \times \mathbf{f}]. \tag{3.37}$$

Substituting the expression (3.37) into (3.11) and solving the resulting equation for $\nabla \times \mathbf{q}$ one finds that

$$\nabla \times \mathbf{q} = -(1-l_2^2\nabla^2)\nabla\hat{\theta} + \frac{\mu}{2\tau}l_1^2\nabla^4\hat{\mathbf{k}} - \frac{1}{2\tau}l_3^2\nabla \times \mathbf{f} - \frac{1}{2\tau}\hat{\mathbf{c}}. \tag{3.38}$$

Taking the curl of (3.34) and subsequently employing the vector identity (2.19), the result

$$\nabla^2\hat{\mathbf{k}} = -\nabla\times\mathbf{h} \tag{3.39}$$

follows, hence from (3.9), (3.38) and (3.39)

$$\hat{\gamma} = l_2^2\nabla^2\nabla\hat{\theta}-\frac{\mu}{2\tau}l_1^2\nabla^2\nabla\times\mathbf{h}-\frac{1}{2\tau}l_3^2\nabla\times\mathbf{f}-\frac{1}{4\tau}\hat{\mathbf{c}}. \tag{3.40}$$

To obtain the differential equation governing $\hat{\theta}$ take the divergence of (3.11) and employ $(3.8)_2$, thus

$$(1-l_2^2\nabla^2)\nabla^2\hat{\theta} = -\frac{1}{4\tau}\nabla\,.\,\hat{\mathbf{c}}. \tag{3.41}$$

Defining

$$\hat{h} = l_2^2\nabla^2\hat{\theta}, \tag{3.42}$$

the expression (3.2) for $\hat{\gamma}$ follows from (3.40) and (3.42) when the notational changes (2.14) have been accomplished. The differential equation (3.5) follows from (3.41) and (3.42). This completes the proof.

There are several alternative methods of proving the validity and completeness of the representation given by (3.1) through (3.5). As an example of alternate method consider the following four steps: The first step consists of uncoupling the system of equations (2.17) and (2.18). Eliminating $\hat{\boldsymbol{\psi}}$ from the system of equations (2.17) and (2.18) by employing $(2.4)_2$, and subsequently performing a series of manipulations, a system of the two uncoupled vector equations in $\mathbf{u}$ and $\hat{\gamma}$ are obtained,

$$(1-l_3^2\nabla^2)[(\lambda+2\mu)\nabla\nabla\,.\,\mathbf{u}+\mathbf{f}]-\mu(1-l_1^2\nabla^2)\nabla\times\nabla\times\mathbf{u}+\tfrac{1}{2}\nabla\times\hat{\mathbf{c}} = 0, \tag{3.43}$$

$$l_2^2\nabla\nabla\,.\,\hat{\gamma}-l_1^2\nabla\times\nabla\times\hat{\gamma}-\hat{\gamma}+\frac{l^2}{2\tau}\nabla\times\mathbf{f}-\frac{1}{4\tau}\hat{\mathbf{c}} = 0. \tag{3.44}$$

The second step consists of finding complete solutions to the uncoupled set of equations (3.43) and (3.44). Equation (3.43) is similar to an equation whose complete solution was obtained by Mindlin [6, Section 13]. A complete solution of (3.44) is easy to find and can be obtained, for example, by appropriately modifying the arguments of Duhem's proof of Clebsch's completeness theorem as given by Sternberg [14, Section 2]. These two complete solutions involve two vector and two scalar potentials. The third step consists of substituting the complete solutions of (3.43) and (3.44) back into the original coupled set of differential equations (2.17) and (2.18). The equations generated in this process will show that the two vector potentials generated in step two are related and one may be eliminated. The fourth and final step consists of using the relationship between the two vector potentials obtained in step three to reduce the results of step two. The resulting equations will again prove the validity and completeness of the representation expressed by (3.1) through (3.5).

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APPENDIX

TABLE OF NOTATIONS

Name	Symbol in the present text	Mindlin [3]	Aero and Kuvshinskii [1,2]	Neuber [4]	Eringen [8]
	Dynamic variables				
Symmetric stress tensor	τ_{ij}	τ_{ij}	$\sigma_{(ij)}$	$t_{(ij)}$	$t_{(ij)}$
Skew-symmetric stress tensor	σ_{ij}	σ_{ij}	$-\sigma_{[ij]}$	$t_{[ij]}$	$t_{[ij]}$
Couple stress tensor	$\hat{\mu}_{ij}$	$\frac{1}{2}e_{jmn}\mu_{imn}$	$\frac{1}{2}\mu_{ji}$	$\frac{1}{2}m_{ij}$	$\frac{1}{2}m_{ij}$
Body force per unit volume	f_i	f_i	ρf_i	P_i	ρf_i
Body couple per unit volume	$\hat{c}_i$	$e_{ijk}\Phi_{[jk]}$	ρm_i	q_i	ρl_i
	Kinematic variables				
Displacement vector	u_i	u_i	u_i	V_i	u_i
Strain tensor	ε_{ij}	ε_{ij}	e_{ij}	$d_{(ij)}$	e_{ij}
Average rotation vector	$\hat{\omega}_i$	$\frac{1}{2}e_{ijk}\omega_{jk}$	$-\omega_i$	$\frac{1}{2}e_{imn}d_{mn}-\omega_i$	$-r_i$
Total rotation vector	$\hat{\psi}_i$	$\frac{1}{2}e_{ijk}\psi_{[jk]}$	Ω_i	ω_i	ϕ_i
Relative rotation vector	$\hat{\gamma}_i$	$\frac{1}{2}e_{ijk}\gamma_{[jk]}$	$\omega_i-\Omega_i$	$-\frac{1}{2}e_{imn}d_{mn}$	$r_i-\phi_i$
Total rotation gradient tensor	$\hat{\kappa}_{ij}$	$\frac{1}{2}e_{jmn}\kappa_{imn}$	r_{ji}	$\omega_{j,i}$	$\phi_{j,i}$
	Material coefficients				
Lamé modulus	λ	λ	λ	$\frac{2v}{1-2v}G$	λ
Lamé shear modulus	μ	μ	μ	G	$\mu+\frac{1}{2}\kappa$
Rotation modulus	τ	β	$-\gamma$	Ga	$\frac{1}{2}\kappa$
A rotation gradient modulus	α	$2\alpha_3$	2η	$4G1^2c$	α
A rotation gradient modulus	η	$\frac{1}{2}\alpha_1$	$\frac{1}{2}\theta$	Gl^2	$\frac{1}{4}\gamma$
A rotation gradient modulus	η'	$\alpha_2-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_3$	$\frac{1}{2}\tau$	Gl^2b	$\frac{1}{4}\beta$

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Абстракт—В работе получается полное решение уравнений равновесия в перемещениях для линейной инфинитизимальной изотропной упругости Коссера. Это решение дается в выражениях функций напряжений, аналогичным функциям Папковича для классической упругости и решениям Миндлина и Тирстена для теории упругости с учетом моментных напряжений.

Предыдущие полные решения этих уравнений, полученные Миндлином и Нейбером, заключают три более скалярных потенциалов, по сравнению с использованным в настоящем решении.